

Definition of Contravariant Velocity Components

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Abstract

This is an old issue in computational fluid dynamics (CFD). What is the so-called contravariant velocity or contravariant velocity component? In the article, we review the basics of tensor analysis and give the contravariant velocity component a rigorous explanation. For a given coordinate system, there exist two uniquely determined sets of base vector systems - one is the covariant and another is the contravariant base vector system. The two base vector systems are reciprocal. The so-called contravariant velocity component is really the contravariant component of a velocity vector for a time-independent coordinate system, or the contravariant component of a relative velocity between fluid and coordinates, for a time-dependent coordinate system. The contravariant velocity components are not physical quantities of the velocity vector. Their magnitudes, dimensions, and associated directions are controlled by their corresponding covariant base vectors. Several 2-D linear examples and 2-D mass-conservation equation are used to illustrate the details of expressing a vector with respect to the covariant and contravariant base vector systems, respectively.

I. Introduction

Written in generalized curvilinear coordinates $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ the 2-D inviscid Navier-Stokes equations are, in a strong conservative form,

$$\vartheta_\tau Q + \vartheta_\xi F + \vartheta_\eta G = 0 \quad (1)$$

$$Q = J^{-1} \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ e \end{pmatrix}, F = J^{-1} \begin{pmatrix} \rho U \\ \rho u U + \xi_x p \\ \rho v U + \xi_y p \\ U(e + p) \end{pmatrix}, G = J^{-1} \begin{pmatrix} \rho V \\ \rho u V + \eta_x p \\ \rho v V + \eta_y p \\ V(e + p) \end{pmatrix},$$

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where, ρ, u, v, p, e are conventional physical properties, J is transformation Jacobian and

$$U = \xi_x u + \xi_y v, \quad V = \eta_x u + \eta_y v \quad (2)$$

Vinokur(Ref. 1) introduced the strong conservative form of Navier-Stokes equations to CFD in 1974, and pointed out that (U, V) were the contravariant components of velocity vector. Compared to the Navier-Stokes equations written in Cartesian coordinates (x, y) , except for the additional inclusion of the transformation Jacobian J , (U, V) are the replacements of velocities (u, v) , and hence Steger (Ref. 2) in 1977 treated (U, V) as velocities and stated that (U, V) were

“ - - - the so-called contravariant velocities along the ξ and η coordinates.”

Since then, the term “contravariant velocities” or “contravariant velocity components” was widely used in computational fluid mechanics and heat transfer, in general without further statement or explanation, (for instances, Refs. 3 - 5). The term “contravariant velocity components” could be interpreted either as, like Vinokur(Refs. 1), “contravariant (velocity) components”, (i.e. contravariant components of velocity) which is strictly a mathematical definition or as, like Steger (Ref. 2), “the components of contravariant velocity” in which the “contravariant velocity” can be loosely defined. In this article “contravariant velocity components” is treated as the latter and is interchangeable with “contravariant velocities”.

Following a similar definition of Ref. 2, while applied for body surface, Ref. 6 explained that

“ - - - the contravariant velocities are the decomposition of velocity vector into components along the ξ coordinate line, U , and along the η coordinate line, V .”

There is some ambiguity in the above explanation. It had never explained the means and procedures of “decomposition” of velocity vector into components. It only described the directions the components associated but had never prescribed its corresponding base vector system. One would intuitively interpret (U, V) as the decomposed physical components of velocity vector along the directions of coordinate lines (ξ, η) . In contrast to Refs. 2 and 6, Ref. 7 stated (for 3-D case) that

“The contravariant velocity components U, V, W are in directions normal to constant ξ, η, ζ surfaces, respectively.”

The above statement contradicts the statements from Refs. 2 and 6 about the directions of contravariant velocity components associated with a coordinate system. As we shall see, neither of the above two statements (from Refs. 6 and 7) is strictly correct. The contravariant velocity components are neither the physical components of a velocity vector nor in directions normal to constant ξ, η, ζ surfaces, respectively. The objective of this article is to clarify what are the contravariant velocity components and to further explain their physical implications.

To avoid confusion, we will start with defining terms based on the algebra of vectors. The term *component* of a vector will be defined. With the definition of covariant and contravariant base vector systems for given coordinates, the general mathematical term “contravariant component” will be discussed and its counter term “covariant component” will also be explained. For simplicity and clarity, we will use several 2-D linear cases as examples. Here in the 2-D examples, the (x^1, x^2) coordinates are interchangeable with (ξ, η) coordinates.

II. Algebra of Vectors

In an n -dimensional vector space a set of n linearly independent vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ is called a *base vector system*. Any vector \mathbf{A} in the space can be expressed as a unique combination of the base vectors $\mathbf{A} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + \dots + c_n\mathbf{e}_n$, or using the summation convention, $\mathbf{A} = c_i\mathbf{e}_i$. The vectors $c_1\mathbf{e}_1, c_2\mathbf{e}_2, \dots, c_n\mathbf{e}_n$ are *component vectors* of vector \mathbf{A} in the $\mathbf{e}_1, \mathbf{e}_2, \dots$, and \mathbf{e}_n directions, respectively. The coefficients, c_1, c_2, \dots , and c_n , are called *components* of vector \mathbf{A} in the $\mathbf{e}_1, \mathbf{e}_2, \dots$, and \mathbf{e}_n directions, respectively, with respect to the base $(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)$.

For example, in Cartesian coordinates, a vector $\mathbf{A} = 5\mathbf{i} + 7\mathbf{j}$ implies

$$\mathbf{e}_1 = \mathbf{i}, \mathbf{e}_2 = \mathbf{j}, \quad c_1 = 5, \quad c_2 = 7.$$

Expressing the same vector in terms of the base vector $((\mathbf{i} - \mathbf{j})/2, \mathbf{j})$ so as $\mathbf{A} = 10(\mathbf{i} -$

$\mathbf{j})/2 + 12\mathbf{j}$, then

$$\mathbf{e}_1 = (\mathbf{i} - \mathbf{j})/2, \mathbf{e}_2 = \mathbf{j}, \quad c_1 = 10, \quad c_2 = 12, \quad (3)$$

or in terms of the base vectors $(2\mathbf{i}, \mathbf{i} + \mathbf{j})$ as $\mathbf{A} = - (2\mathbf{i}) + 7 (\mathbf{i} + \mathbf{j})$, then

$$\mathbf{e}_1 = 2\mathbf{i}, \mathbf{e}_2 = \mathbf{i} + \mathbf{j}, \quad c_1 = -1, \quad c_2 = 7. \quad (4)$$

As illustrated above, the invariant vector \mathbf{A} can be expressed in different base vector systems which may or may not be orthogonal and normalized. While the “component vectors” are physical vectors the magnitude of the components of a vector not only depend on the directions but also depend on the magnitude (and the dimension, discussed later) of the base vector system. Obviously, without the prescription of “base vectors”, the “components” of a vector have no meaning.

III. Contravariant and Covariant Base Vectors

Let x^i be the coordinates of a point and \mathbf{r} be a position vector. As shown in Fig. 1, a covariant base vector, $\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial x^i}$, is tangent to its corresponding coordinate line, x^i , and a contravariant base vector, $\mathbf{g}^i = \nabla x^i$, is normal to the other coordinate lines x^j (or the surface of $x^i = \text{constant}$), $j \neq i$. For any given coordinates x^i , its contravariant and covariant base vectors, \mathbf{g}^i and \mathbf{g}_i are uniquely determined. They have the reciprocal relation,

$$\mathbf{g}^i \cdot \mathbf{g}_j = \delta_j^i = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \quad (5)$$

where δ_j^i is the Kronecker delta. Based on the reciprocal relation, Eq. (5), we have

$$|\mathbf{g}^i| = \frac{1}{|\mathbf{g}_i| \cos(\mathbf{g}^i, \mathbf{g}_i)}, \quad (\text{no summation in } i).$$

If they are orthogonal, then $|\mathbf{g}^i| = \frac{1}{|\mathbf{g}_i|}$ and \mathbf{g}^i and \mathbf{g}_i are in the same direction. Note that, the sketches in Fig. 1 (and Fig. 2) only give a general indication of the directions of \mathbf{g}^i and \mathbf{g}_i and do not imply any thing about their magnitudes.

Indeed, we have $d\mathbf{r} = \mathbf{g}_i dx^i$ and the differential of arc length ds is determined from $ds^2 = d\mathbf{r} \cdot d\mathbf{r}$; therefore

$$ds^2 = \mathbf{g}_i \cdot \mathbf{g}_j dx^i dx^j = g_{ij} dx^i dx^j$$

Here, the entity of g_{ij} thus defined is called the *metric tensor*, because of its being the metric of the space of the coordinates. Similarly, We can define $g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j$. Obviously, g^{ij} is conjugate or reciprocal tensor of g_{ij} , i.e.

$$g^{ik} g_{kj} = \delta_j^i$$

$$(\mathbf{g}^i \cdot \mathbf{g}_j = g^{ik} \mathbf{g}_k \cdot \mathbf{g}_j = g^{ik} g_{kj} = \delta_j^i)$$

The individual g^{ik} and g_{kj} are the components of two fundamental tensors. Let $g = |g_{ij}|$ and $G = |g^{ij}|$ be determinants of g_{ij} and g^{ij} , then $\sqrt{g} = 1/\sqrt{G}$ where \sqrt{g} is the Jacobian of coordinate transformation from old to new and \sqrt{G} is the Jacobian of coordinate transformation from new to old. Details of these relations can be found in regular "Tensor Analysis" books, such as Refs. 8 and 9.

IV. Contravariant and Covariant Components of a Vector

A given vector \mathbf{A} is an invariant, and can be expressed in contravariant and covariant base vector systems as

$$\begin{aligned} \mathbf{A} &= A^1 \mathbf{g}_1 + A^2 \mathbf{g}_2 + A^3 \mathbf{g}_3 - \dots + A^n \mathbf{g}_n \\ &= A_1 \mathbf{g}^1 + A_2 \mathbf{g}^2 + A_3 \mathbf{g}^3 - \dots + A_n \mathbf{g}^n \end{aligned}$$

or, in the summation convention,

$$\mathbf{A} = A^i \mathbf{g}_i = A_i \mathbf{g}^i. \quad (6)$$

Here a super-script index is used for "contravariant" and a subscript index is used for "covariant". According to the definitions in Sect. II, $A^1 \mathbf{g}_1, A^2 \mathbf{g}_2, - \dots, A^n \mathbf{g}_n$ are component vectors of vector \mathbf{A} in the directions of $\mathbf{g}_1, \mathbf{g}_2, - \dots, \mathbf{g}_n$, respectively. Similarly, $A_1 \mathbf{g}^1, A_2 \mathbf{g}^2, - \dots, A_n \mathbf{g}^n$ are component vectors of vector \mathbf{A} in the directions of

$\mathbf{g}^1, \mathbf{g}^2, \dots, \mathbf{g}^n$, respectively. As defined in the previous Section, \mathbf{g}_i and \mathbf{g}^i are two uniquely defined base vector systems for a given coordinate system x^i . (The terms *covariant* and *contravariant* will be explained below.) Hence, we define

A^i as contravariant components of vector \mathbf{A} in the directions of \mathbf{g}_i , with respect to the covariant base vector system $(\mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_n)$.

A_i as covariant components of vector \mathbf{A} in the directions of \mathbf{g}^i , with respect to the contravariant base vector system $(\mathbf{g}^1, \mathbf{g}^2, \dots, \mathbf{g}^n)$.

For simplicity, we call A^i contravariant components of vector \mathbf{A} and A_i covariant components of vector \mathbf{A} , with their association with covariant and contravariant base vectors (respectively) implicitly implied. The reasons why A^i are called contravariant and A_i covariant components of a vector are rooted in fundamentals of tensor theory (Refs. 8 and 9). The essential property of a “vector” (or in general a tensor) is that its components follow the law of coordinate transformation. Setting the base vectors \mathbf{g}_i as the standard of comparison, if in a coordinate transformation, the components are transformed by the same matrix that described the transformation of the base vectors \mathbf{g}_i , the components are called “covariant” and if the components are transformed by the inverse transposed matrix of the the matrix that described the transformation of the base vectors \mathbf{g}_i , the components are called “contravariant”. (The proof of that components A_i are “covariant” and A^i are “contravariant” is beyond the scope of this article.)

One can show that, from Eqs. (5) and (6),

$$A^i = \mathbf{A} \cdot \mathbf{g}^i \text{ and } A_i = \mathbf{A} \cdot \mathbf{g}_i \quad (7)$$

Contravariant and covariant components, A^i, A_i are related to each other

$$A^i = g^{ij} A_j, \text{ and } A_i = g_{ij} A^j$$

To obtain “physical components” of vector \mathbf{A} , one has to decompose the vector with respect to the normalized base vector, so as

$$\mathbf{A} = \hat{A}^i \frac{\mathbf{g}_i}{|\mathbf{g}_i|} = \hat{A}_i \frac{\mathbf{g}^i}{|\mathbf{g}^i|} \text{ (summation on } i\text{)}.$$

Thus the physical components \hat{A}^i along \mathbf{g}_i direction and \hat{A}_i along \mathbf{g}^i direction are

$$\hat{A}^i = |\mathbf{g}_i| A^i = \sqrt{g_{ii}} A^i, \text{ and } \hat{A}_i = |\mathbf{g}^i| A_i = \sqrt{g^{ii}} A_i, \text{ (no sum on } i). \quad (8)$$

Here, the super-script index in component \hat{A}^i and subscript index in \hat{A}_i do not imply “contravariant” and “covariant” as adopted in the article. They don’t follow the law of coordinate transformation, hence they are not components of tensors.

Eq. (7) is useful for evaluation but commonly leads to a mis-interpretation (as in Ref. 7) that A^i is the component of vector \mathbf{A} in the direction of \mathbf{g}^i and A_i the components of vector \mathbf{A} in the direction of \mathbf{g}_i . This mis-interpretation stems from the common use of a dot-product with a unit vector, so-called “projection”, to calculate the component of a vector in the direction of that unit vector. In projection, it implies that a vector is decomposed into two component vectors; one parallel to and another perpendicular to a given direction. (This is different from the physical components mentioned above, as \hat{A}^i calculated in Eq. (8).) In Eq. (7), neither the contravariant base vectors nor the covariant base vectors are normalized, so that Eq. (7) is not a process of projection.

V. 2-D Examples and Continuity Equation

Now in 2-D, let a position vector $\mathbf{r}(\xi, \eta)$ with coordinate system (ξ, η) , then the detailed formula, with relations to the Cartesian quantities, are

$$\mathbf{A} = u \mathbf{i} + v \mathbf{j} = A^1 \mathbf{g}_1 + A^2 \mathbf{g}_2 = A_1 \mathbf{g}^1 + A_2 \mathbf{g}^2,$$

$$\mathbf{g}^1 = \nabla \xi = \xi_x \mathbf{i} + \xi_y \mathbf{j}, \quad \mathbf{g}^2 = \nabla \eta = \eta_x \mathbf{i} + \eta_y \mathbf{j}$$

$$\mathbf{g}_1 = \frac{\partial \mathbf{r}}{\partial \xi} = x_\xi \mathbf{i} + y_\xi \mathbf{j}, \quad \mathbf{g}_2 = \frac{\partial \mathbf{r}}{\partial \eta} = x_\eta \mathbf{i} + y_\eta \mathbf{j}$$

$$\sqrt{g} = x_\xi y_\eta - y_\xi x_\eta$$

$$A^1 = (\mathbf{A} \cdot \mathbf{g}^1) = \xi_x u + \xi_y v, \quad A^2 = (\mathbf{A} \cdot \mathbf{g}^2) = \eta_x u + \eta_y v$$

$$A_1 = (\mathbf{A} \cdot \mathbf{g}_1) = x_\xi u + y_\xi v, \quad A_2 = (\mathbf{A} \cdot \mathbf{g}_2) = x_\eta u + y_\eta v$$

Here A^1 is the contravariant component of \mathbf{A} in the direction of ξ and A^2 is the contravariant component of \mathbf{A} in the direction of η , with respect to the base vector system

$(\frac{\partial \mathbf{r}}{\partial \xi}, \frac{\partial \mathbf{r}}{\partial \eta})$. A_1 is the covariant component of \mathbf{A} in the direction of $\nabla \xi$ and A_2 is the covariant component of \mathbf{A} in the direction of $\nabla \eta$, with respect to the base vector system $(\nabla \xi, \nabla \eta)$.

The relations of contravariant and covariant components to physical quantities can be illustrated in 2-D as Fig. 2. In Fig. 2, the vector $\mathbf{A} = \underline{OC}$ is expressed by physical component vectors $\underline{OB} = (\mathbf{A} \cdot \mathbf{g}^1)\mathbf{g}_1 = A^1 \mathbf{g}_1$ and $\underline{OD} = (\mathbf{A} \cdot \mathbf{g}^2)\mathbf{g}_2 = A^2 \mathbf{g}_2$ in the directions tangent to corresponding coordinate lines, $(\mathbf{g}_1, \mathbf{g}_2)$, or component vectors $\underline{OE} = (\mathbf{A} \cdot \mathbf{g}_1)\mathbf{g}^1 = A_1 \mathbf{g}^1$ and $\underline{OF} = (\mathbf{A} \cdot \mathbf{g}_2)\mathbf{g}^2 = A_2 \mathbf{g}^2$ in directions normal to the other coordinate lines, $(\mathbf{g}^1, \mathbf{g}^2)$, respectively. (Here, a segment with underline, such as \underline{OB} , is a vector and without underline, such as OB , is a scalar.) In Fig. 2, $OBCD$ is a parallelogram formed by $\underline{BC} \parallel \mathbf{g}_2$ and $\underline{CD} \parallel \mathbf{g}_1$, and $OECF$ is a parallelogram formed by $\underline{EC} \parallel \mathbf{g}^2$ and $\underline{CF} \parallel \mathbf{g}^1$. Thus the physical components are, with respect to system $(\mathbf{g}_1, \mathbf{g}_2)$,

$$OB = |\mathbf{g}_1| A^1, \quad OD = |\mathbf{g}_2| A^2,$$

and, with respect to system $(\mathbf{g}^1, \mathbf{g}^2)$,

$$OE = |\mathbf{g}^1| A_1, \quad OF = |\mathbf{g}^2| A_2.$$

It is interesting to see the difference of physical component from the projection of vector in certain direction. In contrast to OB and OD as the physical components, the projections of vector \underline{OC} in directions of covariant base vectors are

$$OM = (\mathbf{A} \cdot \mathbf{g}_1)/|\mathbf{g}_1| = A_1/|\mathbf{g}_1|, \quad ON = (\mathbf{A} \cdot \mathbf{g}_2)/|\mathbf{g}_2| = A_2/|\mathbf{g}_2|.$$

Similarly, the projections of vector \underline{OC} in directions of contravariant base vectors are

$$OG = (\mathbf{A} \cdot \mathbf{g}^1)/|\mathbf{g}^1| = A^1/|\mathbf{g}^1|, \quad OH = (\mathbf{A} \cdot \mathbf{g}^2)/|\mathbf{g}^2| = A^2/|\mathbf{g}^2|. \quad (9)$$

We would like to illustrate the above discussions with several examples, by using the same vector as in Sect. II, $\mathbf{A} = 5\mathbf{i} + 7\mathbf{j}$, expressed in various contravariant and covariant base vectors.

As a first example, consider the coordinate system $\xi = 2x$, $\eta = y$ which is still orthogonal but stretched, thus

$$\mathbf{r}(\xi, \eta) = \xi/2\mathbf{i} + \eta\mathbf{j}$$

$$\mathbf{g}^1 = \xi_x\mathbf{i} + \xi_y\mathbf{j} = 2\mathbf{i}, \quad \mathbf{g}^2 = \eta_x\mathbf{i} + \eta_y\mathbf{j} = \mathbf{j}$$

$$\mathbf{g}_1 = x_\xi\mathbf{i} + y_\xi\mathbf{j} = 1/2\mathbf{i}, \quad \mathbf{g}_2 = x_\eta\mathbf{i} + y_\eta\mathbf{j} = \mathbf{j}$$

Therefore, the contravariant components for the vector \mathbf{A} are

$$A^1 = \mathbf{A} \cdot \mathbf{g}^1 = 10, \quad A^2 = \mathbf{A} \cdot \mathbf{g}^2 = 7$$

and the covariant components are

$$A_1 = \mathbf{A} \cdot \mathbf{g}_1 = 5/2, \quad A_2 = \mathbf{A} \cdot \mathbf{g}_2 = 7.$$

Fig. 3 shows the sketch of this example, with magnitude and direction. Note that the new coordinates (ξ, η) are same as the (x, y) Cartesian coordinates, except ξ is scaled as $2x$. While \mathbf{g}_2 and \mathbf{g}^2 are identical with in both magnitude and direction, \mathbf{g}_1 and \mathbf{g}^1 are in the same direction as x-coordinate, \mathbf{i} , but different in magnitude, and the magnitudes of A^1 and A_1 are scaled accordingly. The shaded area in Fig. 3 (also in Figs. 4a and 5) indicates as the value of \sqrt{g} , Jacobian of transformation from old to new coordinates.

As a second example, consider the coordinate lines (ξ, η) defined by $\xi = 2x$, $\eta = x+y$, so as

$$\mathbf{r}(\xi, \eta) = \xi/2\mathbf{i} + (\eta - \xi/2)\mathbf{j}$$

$$\mathbf{g}^1 = 2\mathbf{i}, \quad \mathbf{g}^2 = (\mathbf{i} + \mathbf{j})$$

$$\mathbf{g}_1 = (\mathbf{i} - \mathbf{j})/2, \quad \mathbf{g}_2 = \mathbf{j}$$

Therefore, the contravariant components for the vector \mathbf{A} are

$$A^1 = 10, \quad A^2 = 12$$

and the covariant components are

$$A_1 = -1, \quad A_2 = 7.$$

These are identical as Eqs. (3) and (4) in Sect. 1, written as

$$\mathbf{A} = A^1 \mathbf{g}_1 + A^2 \mathbf{g}_2 = 10(\mathbf{i} - \mathbf{j})/2 + 12\mathbf{j}$$

$$\mathbf{A} = A_1 \mathbf{g}^1 + A_2 \mathbf{g}^2 = -(2\mathbf{i}) + 7(\mathbf{i} + \mathbf{j})$$

Fig. 4 shows detailed sketches of the base vectors, with magnitude and direction, and relations with (x, y) . Fig. 5 shows the vector \mathbf{A} expressed in its contravariant and covariant base vector systems. Since the magnitudes of base vectors are not normalized, the marked number are scaled to each base vectors accordingly.

It is interesting to note that if vector \mathbf{A} is decomposed into two component vectors so that one is parallel to \mathbf{g}^1 and another is normal to \mathbf{g}^1 , then $\mathbf{A} = 5/\sqrt{2} \frac{(\mathbf{i}-\mathbf{j})}{\sqrt{2}} + 7/2 \frac{(\mathbf{i}+\mathbf{j})}{\sqrt{2}}$. These are the projection of vector \mathbf{A} on to unit vectors $(\mathbf{i}-\mathbf{j})/\sqrt{2}$ and $(\mathbf{i}+\mathbf{j})/\sqrt{2}$, and are quite different from the expression in contravariant or covariant base vector system.

As a third example, let a new coordinate system be $\xi = (x - y)/2$, $\eta = y$, and hence $\mathbf{r}(\xi, \eta) = (2\xi + \eta)\mathbf{i} + \eta\mathbf{j}$. Therefore, the contravariant and covariant base vectors are

$$\mathbf{g}^1 = (\mathbf{i} - \mathbf{j})/2, \quad \mathbf{g}^2 = \mathbf{j}$$

$$\mathbf{g}_1 = 2\mathbf{i}, \quad \mathbf{g}_2 = (\mathbf{i} + \mathbf{j})$$

Hence, again as Eqs. (3) and (4),

$$\mathbf{A} = A_1 \mathbf{g}^1 + A_2 \mathbf{g}^2 = 10(\mathbf{i} - \mathbf{j})/2 + 12\mathbf{j}$$

$$\mathbf{A} = A^1 \mathbf{g}_1 + A^2 \mathbf{g}_2 = -(2\mathbf{i}) + 7(\mathbf{i} + \mathbf{j})$$

Fig. 6 shows the contravariant and covariant base vectors for this new coordinate system. Note that the coordinate system in Fig. 6 is the conjugate or reciprocal of the the coordinate system in Fig. 4. The contravariant base vectors in the 3rd example are covariant base vectors in the 2nd example and the covariant base vectors in the 3rd example are

contravariant base vectors in the 2nd example. Clearly, the vector \mathbf{A} is an invariant and the specification of coordinate system is essential for vector decomposition.

In fluid dynamics, the continuity equation can be written in integral and differential forms as

$$\begin{aligned} \frac{\partial}{\partial t} \int_v \rho \, dv + \int_S \rho \mathbf{W} \cdot \mathbf{n} dS &= 0, \\ \frac{\partial}{\partial t} \rho(g)^{\frac{1}{2}} + \frac{\partial}{\partial \xi} \rho \mathbf{W} \cdot (g)^{\frac{1}{2}} \mathbf{g}^1 + \frac{\partial}{\partial \eta} \rho \mathbf{W} \cdot (g)^{\frac{1}{2}} \mathbf{g}^2 &= 0, \\ \frac{\partial}{\partial t} \rho(g)^{\frac{1}{2}} + \frac{\partial}{\partial \xi} \rho(g)^{\frac{1}{2}} U + \frac{\partial}{\partial \eta} \rho(g)^{\frac{1}{2}} V &= 0. \end{aligned} \tag{10}$$

Here, ρ is density, $\mathbf{W} = u\mathbf{i} + v\mathbf{j}$ the velocity vector and the contravariant velocity components, $U = \mathbf{W} \cdot \mathbf{g}^1$ and $V = \mathbf{W} \cdot \mathbf{g}^2$ are contravariant components of velocity vector \mathbf{W} , in the directions of covariant base vectors \mathbf{g}_1 and \mathbf{g}_2 , respectively. Here \sqrt{g} is the same as J^{-1} in Eq. (1), the Jacobian of transformation from old to new coordinates (or $\sqrt{g}^{-1} = \sqrt{G} = J$). In a finite volume formation, \sqrt{g} is the volume of the grid cell and $(g)^{\frac{1}{2}} \mathbf{g}^1$ and $(g)^{\frac{1}{2}} \mathbf{g}^2$ are surface normal in the directions of \mathbf{g}^1 and \mathbf{g}^2 , respectively. The concept of mass conservation can be interpreted accordingly. While (U, V) are computed by the dot-product of velocity with \mathbf{g}^1 and \mathbf{g}^2 , respectively, it should now be clear that they are neither physical components of velocity vector nor in directions normal to the corresponding surfaces.

It is important to note that the contravariant velocity components, (U, V) , do not have the physical dimension of \mathbf{W} (velocity) itself. In fact, in addition to their magnitudes, the dimensions of contravariant and covariant components are controlled by the dimensions of the covariant and contravariant base vectors, respectively. In Eq. (5) the contravariant and covariant base vectors are reciprocal not only in magnitude but also in dimension. This can be clearly illustrated by writing the continuity equation in cylindrical coordinates as

$$\begin{aligned} \xi = r = \sqrt{(x^2 + y^2)}, \quad \eta = \theta = \tan^{-1} y/x \\ \mathbf{g}_1 = \mathbf{g}^1 = \cos\theta \, \mathbf{i} + \sin\theta \, \mathbf{j}, \quad \sqrt{g} = r, \\ \mathbf{g}_2 = -r (\sin\theta \, \mathbf{i} - \cos\theta \, \mathbf{j}), \quad \mathbf{g}^2 = -1/r (\sin\theta \, \mathbf{i} - \cos\theta \, \mathbf{j}) \end{aligned}$$

$$U = u \cos\theta + v \sin\theta, \quad V = 1/r(v \cos\theta - u \sin\theta)$$

$$\frac{\partial}{\partial t} r\rho + \frac{\partial}{\partial r} r\rho(u \cos\theta + v \sin\theta) + \frac{\partial}{\partial \theta} \rho(v \cos\theta - u \sin\theta) = 0.$$

Here, \mathbf{g}_1 and \mathbf{g}^1 are identical. Even they are same in the direction, (orthogonal to \mathbf{g}_1), \mathbf{g}_2 and \mathbf{g}^2 are different in magnitude and dimension. Obviously, in the above, the dimension of contravariant velocity component U is different from that of contravariant velocity component V .

VI. Time-dependent Coordinates and Different Variations

Up to this point, we have only dealt with the cases where the coordinate system independent of time. Actually, Steger (Ref. 1) and Refs. 5 and 6 did include a general transformation with time as, in 2-D, $\xi = \xi(x, y, t)$, $\eta = \eta(x, y, t)$, $\tau = t$ and defined the contravariant velocity components as

$$U = \xi_t + \xi_x u + \xi_y v = \xi_x(u - x_\tau) + \xi_y(v - y_\tau),$$

$$V = \eta_t + \eta_x u + \eta_y v = \eta_x(u - x_\tau) + \eta_y(v - y_\tau).$$

The above relations can be rewritten as

$$U = \mathbf{W} \cdot \mathbf{g}^1 - \mathbf{w}' \cdot \mathbf{g}^1 = (\mathbf{W} - \mathbf{w}') \cdot \mathbf{g}^1, \quad V = \mathbf{W} \cdot \mathbf{g}^2 - \mathbf{w}' \cdot \mathbf{g}^2 = (\mathbf{W} - \mathbf{w}') \cdot \mathbf{g}^2 \quad (11)$$

where $\mathbf{w}' = \frac{\partial \mathbf{r}}{\partial \tau}$ is the coordinate velocity vector. The additional terms are the contravariant components of the coordinate velocity vector. Indeed, $\mathbf{W} - \mathbf{w}'$ is the relative velocity vector. Therefore, the so-called contravariant velocities are the contravariant components of relative velocities, between the fluid and the coordinates (or locally, the grid lines). Obviously, all the previous discussions still hold for the time-dependent coordinate system.

Finally, we would like to point out that not all the so-called contravariant velocities are defined in the same form as discussed above. For instance, in Refs. 10 and 11, they were called as “contravariant velocities written without normalization”. Refs. 10 and 11 have never clarified what would be the “normalized” contravariant velocities. In Ref. 12 the contravariant velocities were defined as

$$U = \mathbf{W} \cdot \frac{\mathbf{g}^1}{|\mathbf{g}^1|}, \quad V = \mathbf{W} \cdot \frac{\mathbf{g}^2}{|\mathbf{g}^2|} \quad (12)$$

In Ref. 13 the “physical contravariant velocity components” (U, V) were defined as

$$U = \mathbf{W} \cdot |\mathbf{g}_1| \mathbf{g}^1, \quad V = \mathbf{W} \cdot |\mathbf{g}_2| \mathbf{g}^2 \quad (13)$$

In Refs. 14 and 15 by including the Jacobian of transformation, the contravariant velocities were defined as

$$U = \mathbf{W} \cdot (g)^{\frac{1}{2}} \mathbf{g}^1, \quad V = \mathbf{W} \cdot (g)^{\frac{1}{2}} \mathbf{g}^2 \quad (14)$$

The three above definitions are quite different from each other. In Eq. (12), (U, V) are the projections of velocity vector on the directions of \mathbf{g}^1 and \mathbf{g}^2 , respectively, and are the same as OG and OH in Fig. 2 given by Eq. (9). They are not “components” of the velocity vector \mathbf{W} as pointed out previously. In Eq. (13), (U, V) are the physical components of velocity vector along the ξ and η directions, given by Eq. (8), and as OB and OD in Fig. 2, respectively. The (U, V) defined in Eq. (14) are not velocity components either. In a finite-volume formulation, Eq. (14) represent the velocity dot-product with surface normals (in the direction of the contravariant base vectors), i.e. the convective flux rate across the corresponding control surfaces. The dimensions of U and V are still controlled by the base vectors \mathbf{g}^1 and \mathbf{g}^2 respectively. This can be seen in the continuity equation, Eq.(10). Nevertheless, the above three variations of (U, V) are no longer components of a tensor, and so are neither contravariant nor covariant components of a vector. We would rather hold to the original meaning of “contravariant” and maintain the definition of contravariant velocities as Eq. (2) or Eq. (11), without any modification.

VII. Concluding Remarks

In this paper we have reviewed the basics of tensor analysis in an attempt to clarify some misconception regard contravariant and covariant components of a vector as used in fluid dynamics. We have indicated that, for any given coordinate system, there are two uniquely determined reciprocal covariant and contravariant base vector systems. The contravariant components are components of a given vector expressed as a unique combination of the covariant base vector system and vis versa, the covariant components are components of a vector expressed with the contravariant base vector system. *A vector is an invariant. It is the components which are contravariant or covariant, associated with the given*

coordinate system. And the so-called "contravariant velocity" or "contravariant velocity component" is really the contravariant component of the velocity vector. As described in Section of Algebra of Vectors, expressing a vector with a combination of base vector is a decomposition process for a specific base vector system. Hence, the contravariant velocity components are decomposed *components* of velocity vector along the directions of coordinate lines, with respect to the covariant base vector system. However, the contravariant (and covariant) components are not physical quantities. Their magnitudes and dimensions are controlled by their corresponding covariant (and contravariant) base vectors.

For a time-dependent coordinate system, the so-called contravariant velocity components are the contravariant components of the velocity vector minus the contravariant components of the coordinate velocity vector. In other words, the contravariant velocities can be interpreted as the contravariant components of relative velocities, between the fluid and the coordinates (or locally, the grid lines). Obviously, all the discussions hold for both time-dependent and independent coordinate systems.

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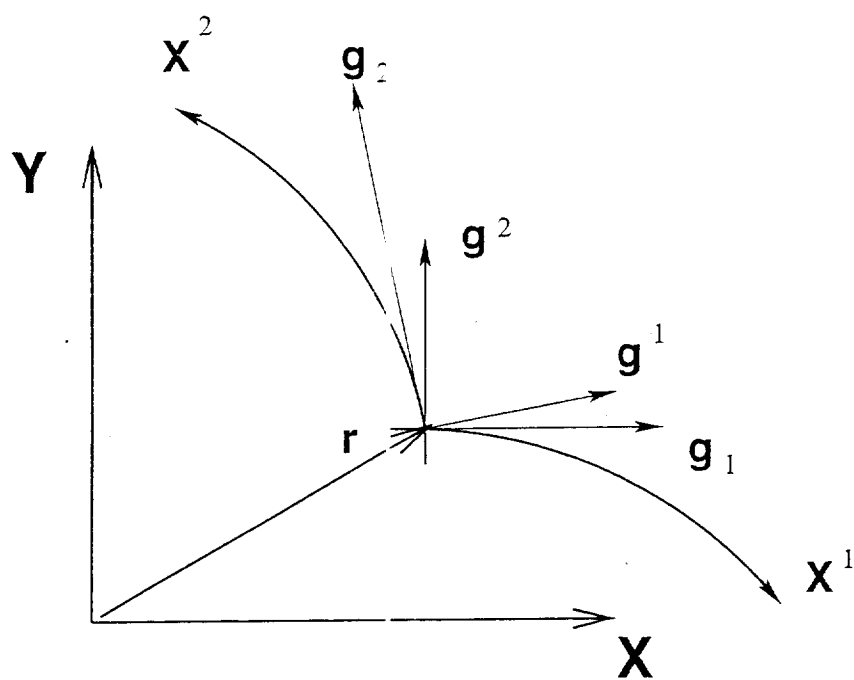
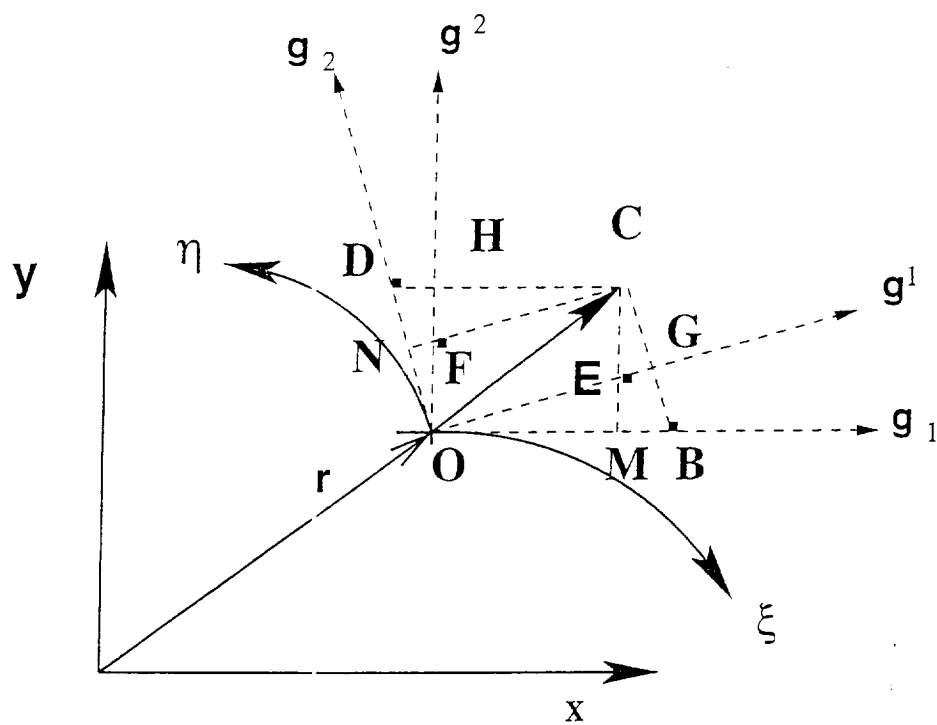


Fig. 1 Sketch of Contravariant and Covariant Base Vectors, g^i and g_i
(indication of direction only).

Fig. 2 Relations of Physical Components and Projections to
Contravariant and Covariant Components



$$A = \underline{OC} = \underline{OB} + \underline{OD} = \underline{OE} + \underline{OF}$$

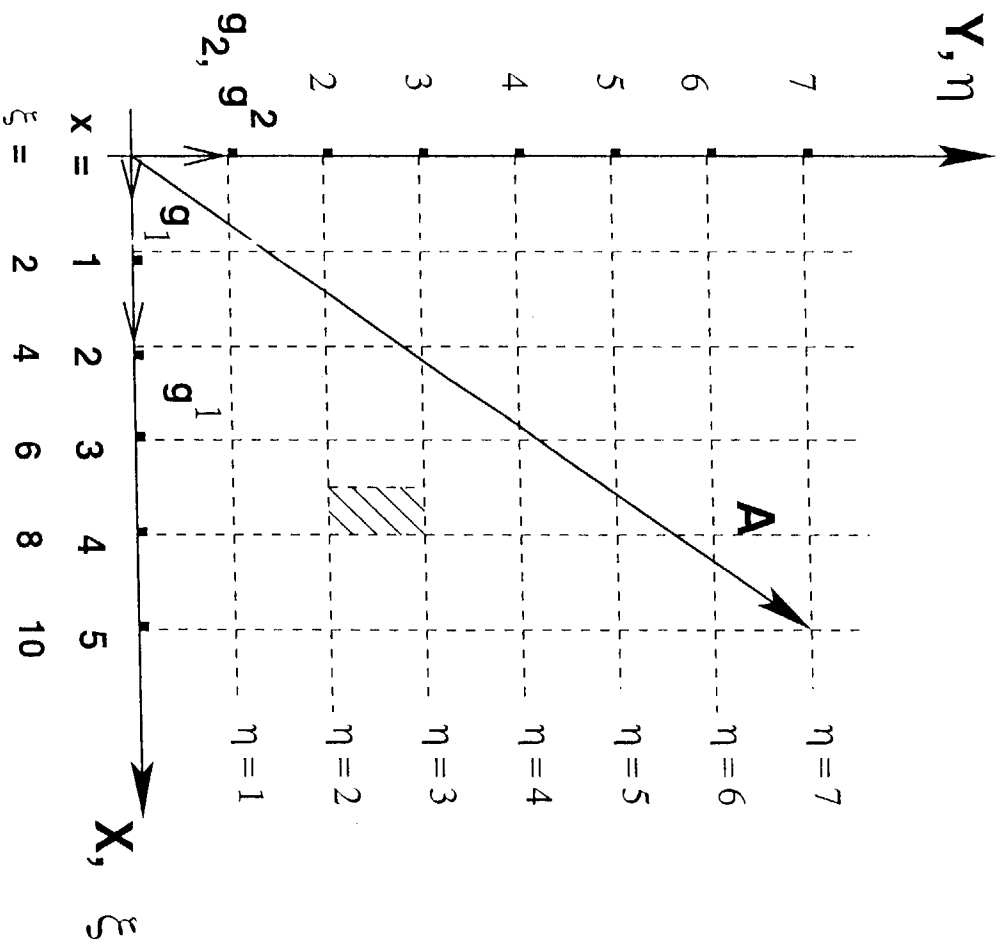
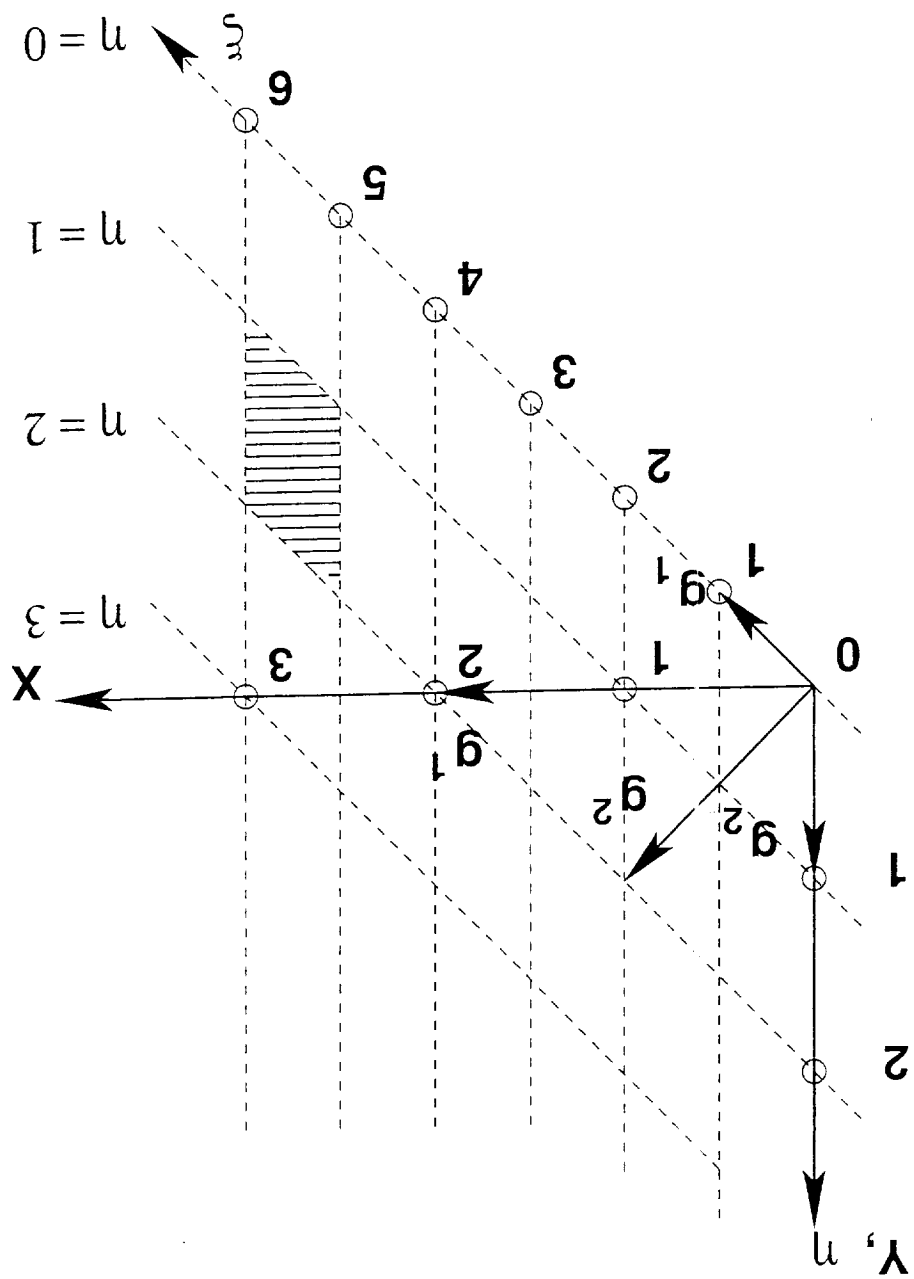


Fig. 3 The Contravariant and Covariant Base Vectors in the Coordinates of $\xi = 2x$ and $\eta = y$, with Vector $A = \xi i + \eta j$

Fig. 4 The Contravariant and Covariant Base Vectors in the Coordinates of $\xi = 2x$ and $\eta = x + y$



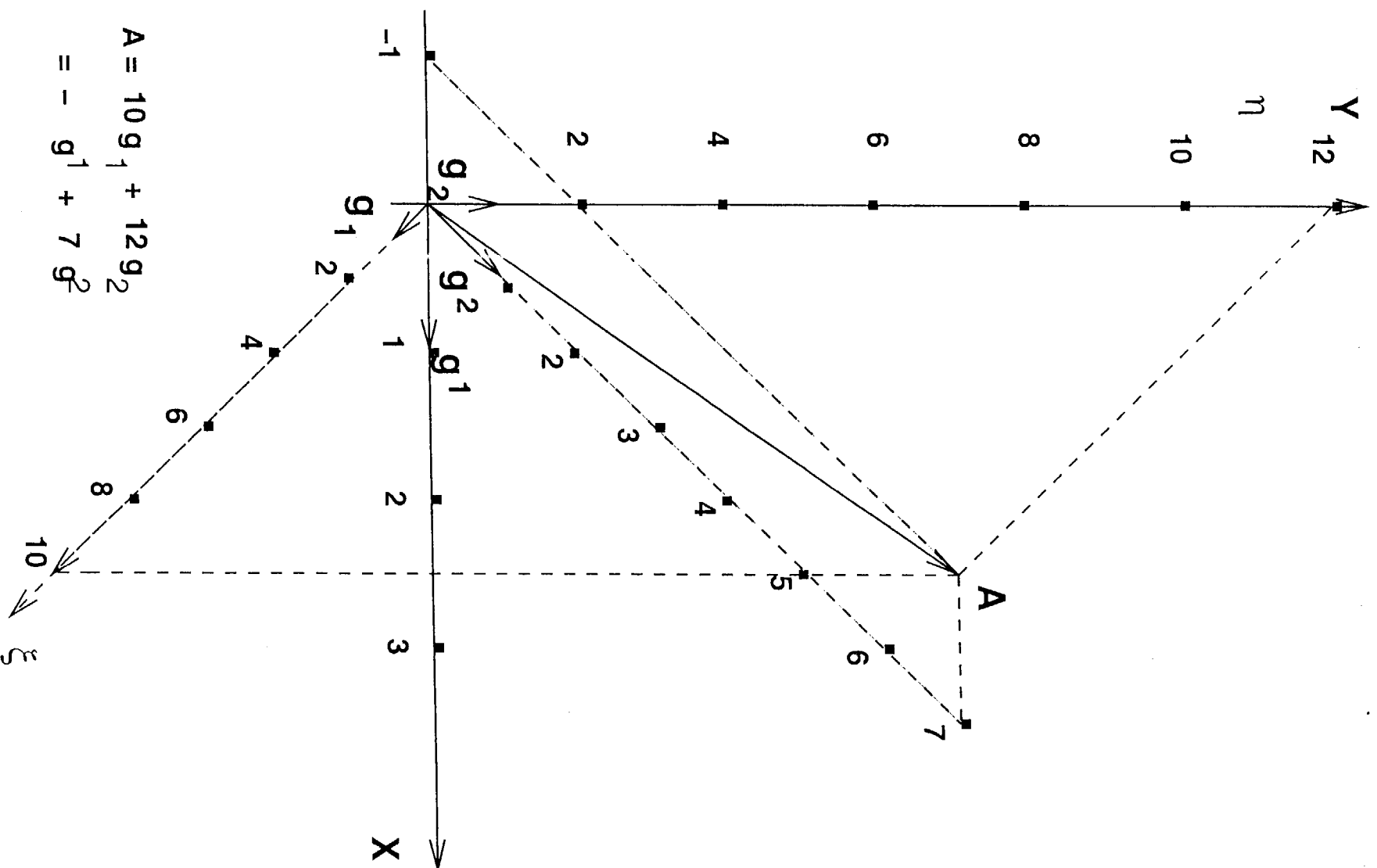


Fig. 5 A Vector $A = 5i + 7j$ Expressed with respect to Contravariant and Covariant Base Vector Systems in the Coordinates of $\xi = 2x$ and $\eta = x + y$

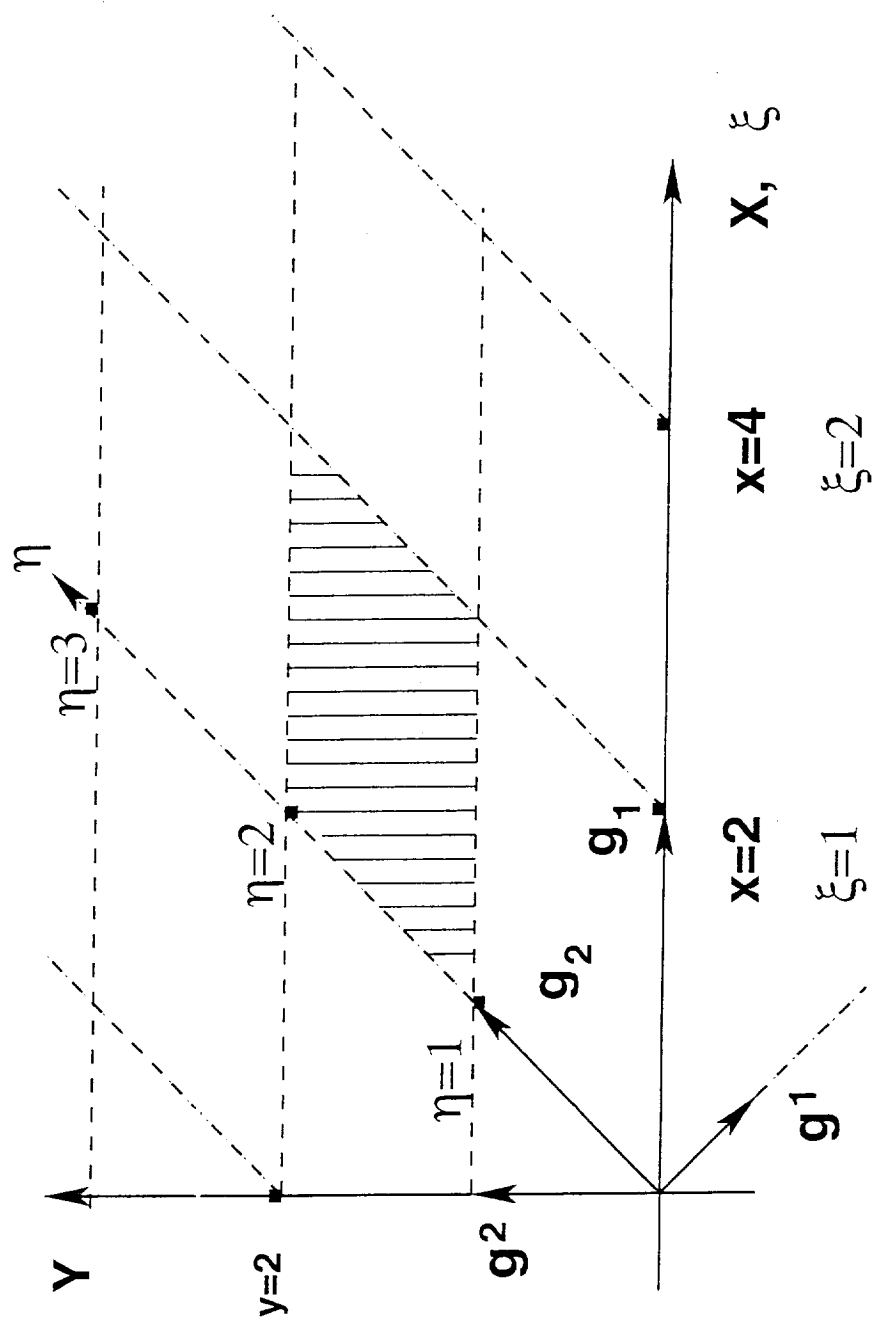


Fig. 6 The Contravariant and Covariant Base Vectors in the Coordinates of $\xi = (x - y)/2$ and $\eta = y$